Mathematical Modeling of Fluid Systems

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Fluid Systems

- As the most versatile medium for transmitting signals and power fluid (gas or liquid) have wide usage in industry. In engineering terms
- **hydraulic** describes fluid systems that use liquids (e.g., oil or water)
- **pneumatic** applies to those using air or other gases.
Why Model Fluid Systems?

• Hydraulic systems is used in machine tool applications, aircraft control systems, where high power to weight ratio, accuracy and quick response is required.

• Industrial processes often involve systems consisting of liquid-filled tanks connected by pipes having orifices, valves, and other flow restricting devices.

• Therefore, it is important to develop a systematic method to mathematically model different types of fluid systems.

Example of Fluid Circuit
Fluid Power Sources

- Pressure Source: Cetrefugal Pump
- Flow Source: Gear Pump

Basic Fluid Elements

There are 3 basic elements in fluid systems:

1. Fluid Capacitor
2. Inertor (long pipe)
3. Fluid Resistor
Fluid Capacitor

A fluid capacitor is an energy storage element, analogous to capacitor in electrical systems.

Fluid Capacitance: \( C = \frac{\text{Change in Stored Fluid Volume}}{\text{Change in Fluid Pressure}} \)

\[
C = \frac{\Delta V_c}{\Delta P_c}
\]

\[
\dot{P}_c = \frac{1}{C} q_c
\]

\( V_c = \text{stored fluid (m}^3\)\)

\( q_c = \text{volumetric flowrate (m}^3/\text{sec}) \)

\( P_c = \text{Pressure (N/m}^3\)\)

Liquid Tank

Flowrate = rate of change of volume

\[
q_c = \dot{V}_c = A \dot{h}_c \Rightarrow \dot{h}_c = \frac{1}{A} q_c
\]

Or equivalently, in terms of pressure

\[
\dot{P}_c = \frac{\rho g}{A} q_c
\]

Capacitance: \( C = A \) (head), \( C = A/\rho g \) (pressure)

A = cross-sectional area (m\(^2\))
**Fluid Inertance**

Fluid inertance is due to fluid inertia such as that in a long pipe. It is defined as

\[
I = \frac{\text{Pressure Differential}}{\text{Rate of Change of Flowrate}}
\]

\[
\dot{q}_I = \frac{1}{I} P_I
\]

Long pipe: \( I = \rho L/A, \ \rho = \text{fluid density}, \ L = \text{pipe length}, \ A = \text{cross-sectional area} \)

If head differential \((h_I = P_I/\rho g)\) is used instead of \(P_I\) then, \( I = L/gA, \ \text{i.e.,} \ \frac{dq_I}{dt} = (gA/L)h_I \)

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**Fluid Resistance**

A fluid resistor is a dissipative fluid element representing a valve or other flow restricting connections.

**Fluid Resistance:**

\[
R = \frac{\text{Change in Pressure (or Head)}}{\text{Change in Flowrate}}
\]

- **Laminar Flow:**
  \( h_R = Rq_R \)

- **Turbulent Flow:**
  \( h_R = f(q_R) \)
  \( R = df(q_R)/dq_R \)
### Mathematical Modeling

1. Identify the basic elements within the system and write the corresponding constitutive relationship (i.e., elemental equation) for each element with respect to a certain equilibrium level (Usually nonzero)

2. Write the applicable interconnection equations, such as pressure or flow balance equations.

3. Assemble the resulting equations to form a system of ordinary differential equations with the same number of unknown variables as the number of equations.

4. Apply Laplace transform if transfer function or further reduction of the equations are required.

### Modeling of Liquid-Level Systems

We will go over the following text examples in class:

1. Example problem on page 187, Fig. 5-2

2. Liquid-level system with interaction, page 189, Fig. 5-5.

3. Liquid-Level System including a long pipe (inertance effect) to be discussed in class (next slide)

4. Linerization of nonlinear systems (section 5-4). Examples 5-3 and 5-4.
Example of Inertance Effect

Derive a diff. Eq. relating the output flow rate $q_o$ to the input flow rate $q_i$.

Tank model: $C \frac{dh_c}{dt} = q_i - q_o$, Valve: $h_o = R q_o$
Long Pipe: $I \frac{dq_o}{dt} = h_c - h_o = h_c - R q_o$, $I$=inertance=$L/gA$
Substituting for $h_c$ from the 2nd eq. Into 1st gives:

$$\frac{d}{dt}(C q_o) + R C \frac{d}{dt} q_o + q_o = q_i$$

Mathematical Modeling of Hydraulic Systems

- Hydraulic systems are used in many real-world systems including machine tool applications, aircraft control systems, automobiles.
- They have high power to weight ratio, and fast starting, stopping, and reversal actions.
- They often operate under a wide pressure range (145psi-5000psi). Power/weight ratio can be increased by increasing operating pressure.
Basic Lump Elements of Hydraulic Systems

- **Resistors**: are used to model pressure drop due to an orifice, friction, leakage, etc. Hydraulic resistors are often nonlinear.

- **Capacitors**: are used to model the compressibility of a hydraulic fluid. Assuming an incompressible fluid, fluid capacitance may be ignored.

- **Inertance**: is often insignificant thus ignored.
Modeling an Orifice

Laminar (low $\Delta p$): $q = \frac{\Delta p}{R}$, $R =$ flow resistance

Turbulent (high $\Delta p$): $q \propto \Delta p^{1/2}$

Power Steering Model
Hydraulic Servomotor/Amplifier

Modeling objective: relate output $y$ to input $x$:

Basic flow-pressure relationship:

- $q_1 = C(p_s - p_1)^{1/2}x$
- $q_2 = C(p_2 - p_0)^{1/2}x$
- $C$: constant

Assuming turbulent (high pressure differential) flow

Simple Model

Assume:

- Incompressible flow $\implies q_1 = q_2 = Ady/dt$
- No fluid leakage

$q_1 = q_2$ also implies that $p_s - p_1 = p_2 - p_0 \implies p_s = p_1 + p_2$

Letting $\Delta p = p_1 - p_2$ then $p_1$ and $p_2$ can be expressed as $p_1 = (p_s + \Delta p)/2$, and $p_2 = (p_s - \Delta p)/2$. Thus

$q_1 = C \sqrt{p_s - p_1} x = \sqrt{\frac{p_s - \Delta p}{2}} x = f(x, \Delta p)$

Linearizing about $\bar{x} = 0$, $\bar{\Delta p} = 0$ gives

$q_1 = K_1 x$, $K_1 = C \sqrt{\frac{p_s}{2}} \implies y = \frac{K_1}{A} x$
A more realistic Model

We assume incompressible fluid \( q_1 = q_2 \) but allow:

- Fluid leakage
- Load inertia and damping

\[ q_1 = q_L + Av \]

\( q_L \) = flow leakage = \( \Delta p/R \),

\( R \) = flow resistance (laminar)

Mechanical subsystem: \( F = A \Delta p = mdv/dt + bv \)

Overall Mathematical Model

Hydraulic subsystem: \( q_1 = \Delta p/R + Av \Rightarrow \Delta p = R(K_1x - Av) \)

Mechanical subsystem: \( F = A \Delta p = mdv/dt + bv \)

Overall input-output model:

\[ mdv/dt + (b + A^2 R)v = ARK_1x \]

Note: As \( R \to \infty \) then \( v \to (K_1/A)x \)

And \( T \to 0 \)